hep-th/0103108

The Cardy-Verlinde Formula and Taub-Bolt-AdS Spacetimes

Danny Birmingham[1]

Department of Mathematical Physics
University College Dublin
Belfield, Dublin 4, Ireland

Soussan Mokhtari[2]

Institute of Technology, Blanchardstown
Blanchardstown Road North
Dublin 15, Ireland

Abstract

We consider the conformal field theory which is dual to the Taub-Bolt-AdS spacetime. It is shown that the Cardy-Verlinde formula for the entropy of the conformal field theory agrees precisely with the entropy of the Taub-Bolt-AdS spacetime, at high temperatures. This result may be viewed as providing a conformal field theory interpretation of Taub-Bolt-AdS entropy.

March 2001

[1]E-mail: dannyb@pop3.ucd.ie
[2]E-mail: susan.mokhtari@itb.ie

In a recent paper [1], a proposal was put forward for the entropy of a D-dimensional conformal field theory on $\mathbf{R} \times S^{D-1}$, with metric

$$ds^2 = -dt^2 + R^2 d\Omega_{D-1}^2. \tag{1}$$

This Cardy-Verlinde formula takes the form

$$S_{\mathrm{CFT}} = \frac{2\pi R}{D-1}\sqrt{E_C(2E - E_C)}, \tag{2}$$

and expresses the CFT entropy in terms of the energy E, the Casimir energy E_C, and the radius R of S^{D-1}. The Casimir energy is defined [1] as the sub-extensive part of the energy E. For CFT in two dimensions, this formula reduces to the standard Cardy formula. In higher dimensions, various aspects of this proposal have been investigated [1]-[12]. In particular, the validity of the formula has been established for the conformal field theories which are dual to the Schwarzschild-AdS black hole [1] and Kerr-AdS black hole [5]. Other cases have been considered in [9].

Within the context of the AdS/CFT correspondence [13]-[15], one can study the boundary conformal field theory at finite temperature defined on a manifold $S^1 \times S^{D-1}$. The relevant AdS configuration in this case is the Euclidean section of the Schwarzschild-AdS black hole [16]. It was shown in [17] that the entropy, energy, and temperature of the boundary CFT can be identified with the corresponding quantities in the black hole spacetime. Furthermore, in the limit of high temperatures, conformal invariance can then be invoked to show that the Bekenstein-Hawking entropy of the Schwarzschild-AdS black hole scales correctly with the horizon volume. However, in order to fix the proportionality constant between the entropy and the horizon volume, one requires additional information, such as a more detailed knowledge of the boundary conformal field theory. The Cardy-Verlinde formula can be viewed as providing this additional information. In this way, one obtains a microscopic (CFT) interpretation of black hole entropy. Alternatively, one can view the Schwarzschild-AdS black hole as a testing ground for the verification of the Cardy-Verlinde formula.

It is clearly of interest to study further examples of spacetimes which have gravitational entropy. One such class is the four-dimensional Taub-NUT-AdS and Taub-Bolt-AdS spacetimes [18]-[22]. These spacetimes have the property that they are only locally asymptotically anti-de Sitter, and are parametrized by an integer k, and a positive real number s. In particular, the boundary CFT is then defined on a non-trivial S^1 bundle over S^2. The $k = 1$ case is of particular interest, as it exhibits a phase transition similar to the Schwarzschild-AdS case [21, 22]. Aspects of the dual CFT have been studied in [23].

In this paper, we present an explicit verification of the Cardy-Verlinde formula for these Taub-NUT and Taub-Bolt spaces. We find that the Cardy-Verlinde formula yields a CFT entropy in precise agreement with the gravitational entropy, in the limit of high temperature. In [21, 22], the correct high temperature dependence of the gravitational entropy was shown to follow from the AdS/CFT correspondence. Thus, the Cardy-Verlinde formula provides the additional information necessary to fix the numerical coefficient in the microscopic (CFT) derivation of the entropy of the Taub-Bolt-AdS spacetime.

The line element of the Taub-NUT-AdS metric can be written in the form [22]

$$\begin{aligned} ds^2 &= \frac{l^2 B}{4}\left[\frac{F(r)}{B(r^2-1)}(d\tau + B^{1/2}\cos\theta\, d\phi)^2 + \frac{4(r^2-1)}{F(r)}dr^2 \right. \\ &+ \left. (r^2-1)(d\theta^2 + \sin^2\theta\, d\phi^2)\right], \end{aligned} \tag{3}$$

where

$$F_{\rm nut}(r) = Br^4 + (4-6B)r^2 + (8B-8)r + (4-3B). \tag{4}$$

Here, B is an arbitrary constant which is related to a 'nut' charge, and $\Lambda = -3/l^2$ is the cosmological constant. The Euclidean time coordinate, τ, has period $\beta = 4\pi B^{1/2}$.

The Taub-Bolt-AdS metric has the same form as (3), with

$$F_{\rm bolt}(r) = Br^4 + (4-6B)r^2 + \left(-Bs^3 + (6B-4)s + \frac{3B-4}{s}\right)r + (4-3B), \tag{5}$$

and

$$B = \frac{2(ks-2)}{3(s^2-1)}. \tag{6}$$

Here, k is the first Chern number of the S^1 bundle over S^2, and s is an arbitrary parameter, which must satisfy the constraints $s > 1$, and $s > 2/k$. Also, $r > s$, and the spacetime has a bolt at $r = s$. The periodicity in Euclidean time is now $\beta = 4\pi B^{1/2}/k$. The parameter s may lie on either an upper branch or lower branch, defined by (6), namely

$$s_\pm = \frac{k}{3B}\left(1 \pm \sqrt{1 - \frac{12B}{k^2} + \frac{9B^2}{k^2}}\right). \tag{7}$$

For our purposes here, we need only recall that the thermodynamics of the Taub-NUT-AdS and Taub-Bolt-AdS spacetimes has been considered in [21, 22]. The crucial point is to decide on an appropriate reference background in order to compute the action of these spaces. An alternative approach without the use of a background has been discussed in [24, 25]. In [21, 22], it was shown that the relevant background is the Taub-NUT-AdS spacetime with k points identified on S^1. The action of the Taub-Bolt-AdS space relative to the Taub-NUT-AdS background is then given by [21, 22]

$$I = -\frac{\pi l^2}{18k}\frac{(ks-2)[k(s^2+2s+3) - 4(2s+1)]}{(s+1)^2}. \tag{8}$$

This leads directly to an expression for the entropy of the Taub-Bolt-AdS space, via $S = \beta\partial_\beta I - I$. This yields

$$S_{\rm AdS} = \frac{\pi l^2}{6}\frac{(ks-2)}{(s+1)^2}\left[(s^2+2s-1) - \frac{4}{k}\right]. \tag{9}$$

Finally, the energy of the spacetime is given by $E = \partial_\beta I$, namely

$$E = \frac{l^2}{36B^{1/2}} \frac{(s-1)(ks-2)[k(s+3)+4]}{(s+1)^2}. \tag{10}$$

As in the Schwarzschild-AdS case, the entropy and energy of the strongly coupled dual CFT can be identified with (9) and (10). In the present case, the three-dimensional dual CFT is defined on a non-trivial S^1 bundle over S^2. However, we should note that the boundary metric (3) takes the form

$$\lim_{r\to\infty} \frac{4}{l^2 B} \frac{R^2}{r^2} ds^2 = R^2(d\tau + B^{1/2}\cos\theta\, d\phi)^2 + R^2(d\theta^2 + \sin^2\theta\, d\phi^2). \tag{11}$$

Thus, in order to coincide with (1), the inverse temperature of the CFT must be re-scaled by a factor of R relative to the inverse temperature of the Taub-Bolt-AdS spacetime. Correspondingly, the energy E of the CFT is re-scaled by a factor of $1/R$ relative to (10). It is now straightforward to compute the Casimir energy of the CFT as defined in [1], namely

$$E_C = DE - (D-1)TS. \tag{12}$$

Here, $\beta = 4\pi B^{1/2} R/k$ is the inverse temperature of the CFT. We find

$$E_C = \frac{l^2}{6R\, B^{1/2}} \frac{(ks-2)(2s-k)}{(s+1)^2}. \tag{13}$$

According to the Cardy-Verlinde formula, the entropy of the CFT is then given by

$$\begin{aligned} S_{\text{CFT}} &= \pi\sqrt{E_C(2E - E_C)} \\ &= \frac{\pi l^2}{6} \frac{(ks-2)}{(s+1)^2} \left[\frac{1}{2}(2s-k)(s+2)(s^2-1)\right]^{1/2}. \end{aligned} \tag{14}$$

It remains to check the relation between this CFT entropy and the entropy of the AdS spacetime given by (9). In the limit of high temperature, $B \to 0$, we find that the bolt size on the upper branch (7) takes the form

$$s_+ = \frac{2k}{3B} - \frac{2}{k} + O(B). \tag{15}$$

In this limit, the energy and Casimir energy of the CFT are given by

$$E = \frac{l^2 k^3}{54R\, B^{3/2}}, \quad E_C = \frac{l^2 k}{3R\, B^{1/2}}. \tag{16}$$

As expected, the Casimir energy is smaller that the energy for high temperature. Furthermore, we observe the expected leading order temperature dependence, namely $E \sim T^3$, and $E_C \sim T$. To leading order, the Cardy-Verlinde formula then yields an entropy of the form

$$S_{\text{CFT}} = \frac{\pi l^2}{6} ks = \frac{\pi l^2 k^2}{9B} = S_{\text{AdS}}. \tag{17}$$

Thus, we see that the Cardy-Verlinde formula is in precise numerical agreement with the high-temperature limit of the gravitational entropy. As predicted from the AdS/CFT correspondence, the entropy scales as β^{-2} [21, 22]. However, we now have the additional information required in order to fix the numerical coefficient precisely. Thus, one can state that the Cardy-Verlinde formula does indeed yield the correct expression for the entropy of the CFT which is dual to the Taub-Bolt-AdS spacetime, at high temperature. However, the full functional form given by (9) and (14) do not agree at sub-leading order. Of course, one should note that in the limit of high temperature, the entropy of the Taub-Bolt-AdS space reduces to the limit of the standard formula $A_{\rm bolt}/4$, where the area of the bolt is given by [22] $A_{\rm bolt} = (2\pi l^2(ks-2)/3)$.

It is worthwhile to consider the $k=1$ case more closely. For temperatures above a critical value, there are two possible Taub-Bolt-AdS spaces. The spacetime corresponding to the larger value of s, discussed above, is thermodynamically stable. However, the space with the smaller value of s is thermodynamically unstable [21, 22]. Nevertheless, it is useful to examine the Cardy-Verlinde formula in this case. In the limit of high temperature on the lower branch, we have

$$s_- = 2 + \frac{9B}{2} + O(B^2). \tag{18}$$

To leading order, the energy and Casimir energy of the CFT are then given by

$$E = \frac{l^2 B^{1/2}}{8R}, \quad E_C = \frac{l^2 B^{1/2}}{4R}. \tag{19}$$

On this branch, the Casimir energy is greater than the energy. Furthermore, to leading order, the AdS entropy and CFT entropy are given by

$$S_{\rm AdS} = \frac{\pi l^2 B}{4}, \quad S_{\rm CFT} = \frac{\pi l^2 B}{2\sqrt{2}}. \tag{20}$$

Thus, we see that the numerical coefficients do not agree in this case. We conclude that for the $k=1$ case, the Cardy-Verlinde formula agrees with the entropy of the thermodynamically stable Taub-Bolt-AdS spacetime, in the limit of high temperature.

In conclusion, we have verified the Cardy-Verlinde formula for the entropy of CFT defined on non-trivial S^1 bundles over S^2. This is achieved by considering the corresponding dual spaces, which are given by the Taub-NUT-AdS and Taub-Bolt-AdS metrics. In the limit of high temperature, we have shown that indeed the Cardy-Verlinde formula for the CFT entropy agrees precisely with the gravitational entropy. Thus, we achieve both a CFT derivation of the Taub-Bolt-AdS entropy, and a verification of the Cardy-Verlinde formula.

References

[1] E. Verlinde, *On the Holographic Principle in a Radiation Dominated Universe*, hep-th/0008140.

[2] D. Kutasov and F. Larsen, JHEP 0101 (2001) 001.

[3] F.-L. Lin, Phys. Rev. D63 (2001) 064026.

[4] S. Nojiri and S. Odintsov, *AdS/CFT Correspondence, Conformal Anomaly and Quantum Corrected Entropy Bounds*, hep-th/0011115.

[5] D. Klemm, A.C. Petkou, and G. Siopsis, Entropy Bounds, Monotonicity Properties and Scaling in CFTs, hep-th/0101076.

[6] B. Wang, E. Abdalla, and R.-K. Su, *Relating Friedmann Equation to Cardy Formula in Universes with Cosmological Constant*, hep-th/0101073.

[7] R. Brustein, S. Foffa and G. Veneziano, *CFT, Holography, and Causal Entropy Bound*, hep-th/0101083.

[8] I. Savonije and E. Verlinde, *CFT and Entropy on the Brane*, hep-th/0102042.

[9] R.-G. Cai, *The Cardy-Verlinde Formula and AdS Black Holes*, hep-th/0102113.

[10] A.K. Biswas and S. Mukherji, *Holography and Stiff-Matter on the Brane*, hep-th/0102138.

[11] Y.S. Myung, *Radially Infalling Brane and Moving Domain Wall in the Brane Cosmology*, hep-th/0102184.

[12] S. Nojiri and S.D. Odintsov, *AdS/CFT and Quantum-Corrected Brane Entropy*, hep-th/0103078.

[13] J. Maldacena, Adv. Theor. Math. Phys. 2 (1998) 231; hep-th/9711200.

[14] S.S. Gubser, I.R. Klebanov, and A.M. Polyakov, Phys. Lett. B428 (1998) 105; hep-th/9802109.

[15] E. Witten, Adv. Theor. Math. Phys. 2 (1998) 253; hep-th/9802150.

[16] S.W. Hawking and D.N. Page, Comm. Math. Phys. 87 (1983) 577.

[17] E. Witten, Adv. Theor. Math. Phys. 2 (1998) 505; hep-th/9803131.

[18] D.N. Page and C.N. Pope, Class. Quantum Grav. 3 (1986) 249.

[19] D.N. Page and C.N. Pope, Class. Quantum Grav. 4 (1987) 213.

[20] D. Kramer, H. Stephani, M. MacCallum, and E. Herlt, *Exact Solutions of Einstein's Field Equations*, Cambridge University Press, 1980.

[21] A. Chamblin, R. Emparan, C.V. Johnson, and R.C. Myers, Phys. Rev. D59 (1999) 064010; hep-th/9808177.

[22] S.W. Hawking, C.J. Hunter, and D.N. Page, Phys. Rev. D59 (1999) 044033; hep-th/9809035.

[23] J.S. Dowker, Class. Quantum Grav. 16 (1999) 1937; hep-th/9812202.

[24] R. Emparan, C.V. Johnson, and R.C. Myers, Phys. Rev. D60 (1999) 104001; hep-th/9903238.

[25] R.B. Mann, Phys. Rev. D60 (1999) 104047; hep-th/9903229.